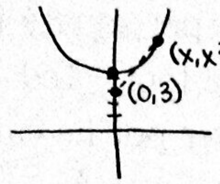
12. Find the point(s) on $f(x) = x^2 + 4$ closest to the point $(0, 3)$.

① 

Guess: $(0,4)$

② minimize $d = \sqrt{(x-0)^2 + (x^2+4-3)^2}$

$= \sqrt{x^2 + (x^2+1)^2}$

OR (trick)

minimize $D = x^2 + (x^2+1)^2$

③ x can be any number

④ ✓

⑤ $D' = 2x + 2(x^2+1)(2x)$

$= 2x(1 + 2(x^2+1))$

$= 2x(1 + 2x^2 + 2)$

$= 2x(2x^2+3) \overset{\text{set}}{=} 0$

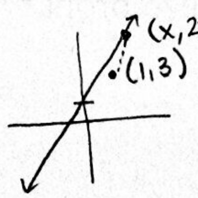
$x = 0$, $\cancel{2x^2+3=0}$ never true

1st Derivative Test:

D'	$-$		$+$
x		0	

⑥ $\boxed{(0,4)}$

13. Find the shortest distance from $y = 2x + 1$ to the point $(1, 3)$.

① 

② minimize $d = \sqrt{(x-1)^2 + (2x+1-3)^2}$

OR $D = (x-1)^2 + (2x-2)^2$

③ x can be any number

④ ✓

⑤ $D' = 2(x-1) + 2(2x-2)$

$= 2x - 2 + 4x - 4$

$= 6x - 6 \overset{\text{set}}{=} 0$

CV: $x = 1$

2nd Derivative Test:

$D'' = 6 > 0$ ∪

⑥ $d = \sqrt{(1-1)^2 + (2 \cdot 1 - 2)^2} = \boxed{0}$

(since $(1,3)$ is on the line!)

14. Ted has decided to make a cylindrical flower pot out of two types of metal. The metal for the sides costs \$15 per square foot and the metal for the base costs \$10 per square foot. Ted has \$30. What are the dimensions of the largest flower pot he can build?

① 

Note: $A_{sides} = 2\pi r h$

$A_{base} = \pi r^2$

Cost = Cost/ft² · Area

② Maximize $V = \pi r^2 h$

③ $r, h > 0$

④ Constraint: $15(2\pi r h) + 10(\pi r^2) = 30$

Solve for h: $30\pi r h = 30 - 10\pi r^2$

$h = \frac{30 - 10\pi r^2}{30\pi r}$

Rewrite V: $V = \cancel{\pi} r^{\cancel{2}} \left(\frac{30 - 10\pi r^2}{30\cancel{\pi}\cancel{r}}\right) = \frac{30r - 10\pi r^3}{30}$

⑤ $V' = \frac{30 - 30\pi r^2}{30}$

$= 1 - \pi r^2 \overset{set}{=} 0$

$\pi r^2 = 1$

$r^2 = 1/\pi$

$r = \pm\sqrt{1/\pi} = 1/\sqrt{\pi}$

2nd Derivative Test:

$V'' = -2\pi r < 0$ when $r = 1/\sqrt{\pi}$

⑥ $r = \boxed{1/\sqrt{\pi}\ \text{ft}}$, $h = \frac{30 - 10\pi(1/\pi)}{30\pi(1/\sqrt{\pi})}$

$= \boxed{\frac{2\sqrt{\pi}}{3}}$

15. A company estimates it will sell $q = 1500 - 300p$ units if each unit costs p dollars. If it costs them 1 dollar to make each unit, how many units should they make to maximize profit?

Revenue = Selling price × Units sold

Cost = Cost per unit × Units sold

Profit = Revenue - Cost

= (Selling price - Cost per unit) × Units sold

① ✓

② Maximize $P = (p-1)(1500 - 300p)$

③ $p > 0$

④ ✓

⑤ $P' = 1(1500 - 300p) + (p-1)(-300)$

$= 1500 - 300p - 300p + 300$

$= 1800 - 600p \overset{set}{=} 0$

$600p = 1800$

$p = 3$

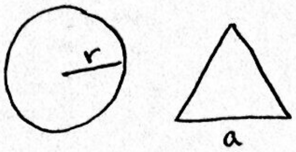
2nd Derivative Test:

$P'' = -600 < 0$ ⌒

⑥ $q = 1500 - 300(3)$

$= \boxed{600 \text{ units}}$

16. The sum of the perimeters of a circle and equilateral triangle is 6 feet. What radius of the circle maximizes the total area? (Hint: The area of a equilateral triangle is $A = \frac{\sqrt{3}}{4}a^2$.)

① 

② maximize $A = \pi r^2 + \frac{\sqrt{3}}{4}a^2$

③ $r, a \geq 0$ ← one could be $= 0$!

④ Constraint: $2\pi r + 3a = 6$

Solve for a: $3a = 6 - 2\pi r$

$a = 2 - \frac{2}{3}\pi r$

Rewrite A: $A = \pi r^2 + \frac{\sqrt{3}}{4}\left(2 - \frac{2}{3}\pi r\right)^2$

⑤ $A' = 2\pi r + \frac{\sqrt{3}}{\cancel{4}}(\cancel{2})\left(2 - \frac{2}{3}\pi r\right)\left(-\frac{\cancel{2}}{3}\pi\right)$

$= 2\pi r - \frac{\sqrt{3}\pi}{3}\left(2 - \frac{2}{3}\pi r\right)$

$= 2\pi r - \frac{2\sqrt{3}}{3}\pi + \frac{2\sqrt{3}}{9}\pi^2 r \overset{\text{set}}{=} 0$

$2\pi r + \frac{2\sqrt{3}}{9}\pi^2 r = \frac{2\sqrt{3}}{3}\pi$

$\left(2\pi + \frac{2\sqrt{3}}{9}\pi^2\right)r = \frac{2\sqrt{3}}{3}\pi$

$r = \dfrac{\frac{2\sqrt{3}\pi}{3}}{2\pi + \frac{2\sqrt{3}\pi^2}{9}} \cdot \frac{9}{9} = \frac{6\sqrt{3}\pi}{18\pi + 2\sqrt{3}\pi^2} \cdot \frac{1/2\pi}{1/2\pi} = \frac{3\sqrt{3}}{9 + \sqrt{3}\pi}$

2nd Derivative Test:

$A'' = 2\pi + \frac{2\sqrt{3}}{9}\pi^2 > 0$ ∪ ← This is a min, not a max!

Look closer at bounds on r:

$0 \leq r \leq \frac{6}{2\pi} = \frac{3}{\pi}$

r	0	$\frac{3\sqrt{3}}{9+\sqrt{3}\pi}$	$3/\pi$
A	1.732	1.079	2.86

↑ max at $r = 3/\pi$ (and $a = 0$)

⑥ $\boxed{r = 3/\pi \text{ ft}}$